2. The number -2 is an eigenvalue of A if and only if the equation $A\mathbf{x} = -2\mathbf{x}$ has a nontrivial solution. This equation is equivalent to $(A + 2I)\mathbf{x} = 0$. Compute

$$A + 2I = \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}$$

The columns of A are obviously linearly dependent, so $(A + 2I)\mathbf{x} = 0$ has a nontrivial solution, and so -2 is an eigenvalue of A.

14. For $\lambda = -2$: $A - (-2I) = A + 2I = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -3 & 0 \\ 4 & -13 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 & -1 \\ 1 & -1 & 0 \\ 4 & -13 & 3 \end{bmatrix}$.

The augmented matrix for $[A - (-2)I]\mathbf{x} = \mathbf{0}$, or $(A + 2I)\mathbf{x} = \mathbf{0}$, is

$$[(A + 2I) \quad \mathbf{0}] = \begin{bmatrix} 3 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 4 & -13 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/3 & 0 \\ 0 & 1 & -1/3 & 0 \\ 0 & -13 & 13/3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/3 & 0 \\ 0 & 1 & -1/3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus $x_1 = (1/3)x_3, x_2 = (1/3)x_3$, with x_3 free. The general solution of $(A + 2I)\mathbf{x} = \mathbf{0}$ is $x_3 \begin{bmatrix} 1/3 \\ 1/3 \\ 1 \end{bmatrix}$.

A basis for the eigenspace corresponding to -2 is $\begin{bmatrix} 1/3 \\ 1/3 \\ 1 \end{bmatrix}$; another is $\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$.

6. Is $A\mathbf{x}$ a multiple of $\mathbf{x}$? Compute $\begin{bmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix} = (-2) \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ So $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ is an eigenvector of

A for the eigenvalue -2.

10. For $\lambda = 4$: $A - 4I = \begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 6 & -9 \\ 4 & -6 \end{bmatrix}$.

The augmented matrix for $(A - 4I)\mathbf{x} = \mathbf{0}$ is $\begin{bmatrix} 6 & -9 & 0 \\ 4 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -9/6 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Thus $x_1 = (3/2)x_2$ and

x_2 is free. The general solution is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (3/2)x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 3/2 \\ 1 \end{bmatrix}$. A basis for the eigenspace corresponding

to 4 is $\begin{bmatrix} 3/2 \\ 1 \end{bmatrix}$. Another choice is $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

18. The eigenvalues of $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -3 \end{bmatrix}$ are 4, 0, and -3, on the main diagonal, by Theorem 1.

24. A simple example of a 2×2 matrix with only one distinct eigenvalue is a triangular matrix with the same number on the diagonal. By experimentation, one finds that if such a matrix is actually a diagonal matrix then the eigenspace is two dimensional, and otherwise the eigenspace is only one dimensional.

Examples: $\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$ and $\begin{bmatrix} 4 & 5 \\ 0 & 4 \end{bmatrix}$.

27. Use the *Hint* in the text to write, for any $\lambda, (A - \lambda I)^T = A^T - (\lambda I)^T = A^T - \lambda I$. Since $(A - \lambda I)^T$ is invertible if and only if $A - \lambda I$ is invertible (by Theorem 6(c) in Section 2.2), it follows that $A^T - \lambda I$ is *not* invertible if and only if $A - \lambda I$ is *not* invertible. That is, λ is an eigenvalue of A^T if and only if λ is an eigenvalue of A.

Note: If you discuss Exercise 27, you might ask students on a test to show that A and A^T have the same characteristic polynomial (discussed in Section 5.2). Since $\det A = \det A^T$, for any square matrix A,

$$\det(A - \lambda I) = \det(A - \lambda I)^T = \det(A^T - (\lambda I)^T) = \det(A^T - \lambda I).$$

28. If A is lower triangular, then A^T is upper triangular and has the same diagonal entries as A. Hence, by the part of Theorem 1 already proved in the text, these diagonal entries are eigenvalues of A^T. By Exercise 27, they are also eigenvalues of A.

29. Let $\mathbf{v}$ be the vector in $\mathbf{R}^n$ whose entries are all ones. Then $A\mathbf{v} = s\mathbf{v}$.

30. Suppose the column sums of an $n \times n$ matrix A all equal the same number s. By Exercise 29 applied to A^T in place of A, the number s is an eigenvalue of A^T. By Exercise 27, s is an eigenvalue of A.

34. You could try to write $\mathbf{x}_0$ as linear combination of eigenvectors, $\mathbf{v}_1, ..., \mathbf{v}_p$. If $\lambda_1, ..., \lambda_p$ are corresponding eigenvalues, and if $\mathbf{x}_0 = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$, then you could *define*

$$\mathbf{x}_k = c_1\lambda_1^k \mathbf{v}_1 + \cdots + c_p\lambda_p^k\mathbf{v}_p$$

In this case, for $k = 0, 1, 2, ...,$

$$\begin{aligned} A\mathbf{x}_k &= A(c_1\lambda_1^k \mathbf{v}_1 + \cdots + c_p\lambda_p^k\mathbf{v}_p) \\ &= c_1\lambda_1^k A\mathbf{v}_1 + \cdots + c_p\lambda_p^k A\mathbf{v}_p \quad \text{Linearity} \\ &= c_1\lambda_1^{k+1}\mathbf{v}_1 + \cdots + c_p\lambda_p^{k+1}\mathbf{v}_p \quad \text{The } \mathbf{v}_i \text{ are eigenvectors.} \\ &= \mathbf{x}_{k+1} \end{aligned}$$